Multidimensional Linear Functions

Friday, February 26, 2021 8:12 AM

Vectors in n-dimensions are the set $\mathbb{R}^n$ = set n-tuples of real #s.
$\mathbb{R}$ = set of scalars

Write
$\vec{x} = (x_1, x_2, \ldots, x_n)$
$\vec{v} = (v_1, v_2, \ldots, v_n)$

Key Operations
1. Addition for $\vec{x}, \vec{y} \in \mathbb{R}$
$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
2. Scalar Mult for $a \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}$
$a\vec{x} = (ax_1, ax_2, \ldots, ax_n)$
3. Dot prod $\vec{x}, \vec{y} \in \mathbb{R}$
$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \sum_{i=1}^{n} x_i y_i$$

- Dot product is bilinear (distributive), $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$, is positive if $\vec{x} \neq \vec{0}$

Cauchy-Schwarz
$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|$

Triangle Inequality
$\|\vec{x} + y\| \leq \|\vec{x}\| + \|\vec{y}\|$

Coordinates
- $x_1, x_2, \ldots, x_n$ are coords of $\vec{x}$
- x_i = ith coord = $\vec{e}_i \cdot \vec{x}$
  where e_i is the ith coord vector
  $\vec{e}_1 = (1, 0, 0, \ldots, 0)$
- $\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i$
  → every vector is a linear combo of the coord vectors $\vec{e}_i$

Linear Functions

e.g. $\mathbb{R}^4 \to \mathbb{R}^7$ $\mathbb{R}^7 \to \mathbb{R}^2$ etc

Idea: the derivative of f at $(x_0, y_0 = f(x_0))$ is given by the linear fcn that best approximates f near (x_0, y_0)
i.e. $y = f'(x_0)(x - x_0) + y_0$
- multiplication by $(f'(x_0))$ is linear part of the fcn
- the $(-x_0)$ and $(+y_0)$ are translations
- in general, when you combine translation w/a linear fcn, you get an affine fcn
  → technically $[y = f'(x_0)(x - x_0) + y_0]$ is affine and the linear part is mult by $[f'(x_0)]$

  affine: $2x + 3 = y$
  linear: $2x = y$ (this is also affine)

In two dimensions
- given $z = f(x, y)$, the best affine fcn that approximates f near $(x_0, y_0, z_0 = f(x_0, y_0))$ is:

$$z = \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + z_0$$

$a := \frac{\partial f}{\partial x}(x_0, y_0)$ $b := \frac{\partial f}{\partial y}(x_0, y_0)$

$$z = \underbrace{(ax + by)}_{\text{linear part}} + \underbrace{(z_0 - ax_0 - by_0)}_{\text{translation}}$$

affine

Note: the translation is just to ensure that the "linear approximation to f" goes through the point (x_0, y_0, z_0)

The derivative is contained in the linear part

Examples of Linear Fcns

$x \mapsto ax$ ("maps to") — the function sending input x to output ax

$\mathbb{R}^1 \to \mathbb{R}^1$

"source" "domain" $\longrightarrow$ "range of possible values" "codomain"

$(x, y) \longmapsto ax + by$ — source is $\mathbb{R}^2$, target domain is $\mathbb{R}^1$; input elements of $\mathbb{R}^2$ and outputs are in $\mathbb{R}^1$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^1$

$(x, y, z) \longmapsto ax + by + cz$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$(x, y) \longmapsto (ax + by, cx + dy)$

$\mathbb{R}^2 \to \mathbb{R}^2$

Linear Fcns

Def

a function $f : \underset{\text{source}}{\mathbb{R}^n} \longrightarrow \underset{\text{target}}{\mathbb{R}^p}$ is linear if

1. for $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$f(\underbrace{\vec{x} + \vec{y}}_{\text{addition in } \mathbb{R}^n}) = \underbrace{f(\vec{x}) + f(\vec{y})}_{\text{addition in } \mathbb{R}^p}$$

2. for $\vec{x} \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$f(\underbrace{a\vec{x}}_{\text{scalar mult in } \mathbb{R}^n}) = \underbrace{a f(\vec{x})}_{\text{scalar mult in } \mathbb{R}^p}$$

Conclusions (what happens if f is linear)

- For $a, b, \vec{x}, \vec{y}$ we have:
$$f(a\vec{x} + b\vec{y}) = f(a\vec{x}) + f(b\vec{y}) = a f(\vec{x}) + b f(\vec{y})$$
$\longrightarrow$ respects linear combos

- For any positive integer m and $\vec{x}_1, \vec{x}_2, \dots \vec{x}_n \in \mathbb{R}^n$ and $a_1, a_2, \dots, a_m \in \mathbb{R}$
$$f(a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_m\vec{x}_m) = f\left(\sum_{i=1}^{m} a_i \vec{x}_i\right)$$
$$= a_1 f(\vec{x}_1) + a_2 f(\vec{x}_2) + \dots + a_m f(\vec{x}_m)$$
$$= \sum_{i=1}^{m} a_i f(\vec{x}_i)$$

Q/ Given f and $\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ how do we find $f(\vec{v})$? (in terms of coords of $\vec{v}$)

A/ $f(\vec{v}) = f\left(\sum_{i=1}^{n} v_i \vec{e}_i\right)$

$$= \sum_{i=1}^{n} v_i F(\vec{e}_i)$$

So if we know $v_1, \ldots, v_n$ and $F(\vec{e}_1), F(\vec{e}_2), \ldots, F(\vec{e}_n)$, then we can find $F(\vec{v})$

Recall

each $F(e_i)$ is a vector in $\mathbb{R}^p$

Idea: F is specified by a collection of n vectors in $\mathbb{R}^p$

→ ie, if you know those n vectors and you know that F is linear, then you know F.

In Fact, given any collection n vectors in $\mathbb{R}^p$ (call them $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathbb{R}^p$) then we can find a linear Fcn:

$$F: \mathbb{R}^n \to \mathbb{R}^p$$

s.t. $F(\vec{e}_i) = \vec{x}_i$ for $i = 1, \ldots, n$

→ This tells us there is a one-to-one correspondence (bijection) between

linear Fcns from $\mathbb{R}^n$ to $\mathbb{R}^p$ AND collections of n vectors in $\mathbb{R}^p$

In terms of coords

$\vec{x}_1 = F(\vec{e}_1) = (a_{11}, a_{21}, \ldots, a_{p1})$

$\vec{x}_j = F(\vec{e}_j) = (a_{1j}, a_{2j}, \ldots, a_{pj})$

↳ we assoc. the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{bmatrix} \longrightarrow \text{represents the linear Fcn } F$$

- each $F(\vec{e}_j)$ is a column vector in this matrix
- matrix has p rows & n columns

Rows

correspond to coords of target $\mathbb{R}^p$

Columns

correspond to coords of domain $\mathbb{R}^n$

Q/ Given M, how to evaluate $F(\vec{v})$ for $\vec{v} = (v_1, \ldots, v_n)$?

A/ We can derive a formula using the fact that F is linear

$$F(\vec{v}) = \sum_{j=1}^{n} v_j F(\vec{e}_j)$$

$$= \sum_{j=1}^{n} v_j (a_{1j}, a_{2j}, \ldots, a_{pj})$$

$$= \sum_{j=1}^{n} (v_j a_{1j}, v_j a_{2j}, \ldots, v_j a_{pj})$$

$$= \sum_{j=1}^{n} v_j a_{1j}, \sum_{j=1}^{n} v_j a_{2j}, \ldots \sum_{j=1}^{n} v_j a_{pj}$$

Conclusion

the ith coord of $f(\vec{v})$ is $\sum_{j=1}^{n} a_{ij} v_j$

This is $M \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ (matrix mult)

There's a natural 1-1 correspondence btw linear fns $\mathbb{R}^n$ to $\mathbb{R}^p$ and $p \times n$ matrices w/ real coefficients.